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NEW RESULTS ON MEASURES OF MAXIMAL ENTROPY

BY

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ABSTRACT

It has recently been demonstrated that there are strongly irreducible subshifts of finite type with more than one measure of maximal entropy. Here we obtain a number of results concerning the uniqueness of the measure of maximal entropy. In addition, we construct for any $d \geq 2$ and k a strongly irreducible subshift of finite type in d dimensions with exactly k ergodic (extremal) measures of maximal entropy. For $d \geq 3$, we construct a strongly irreducible subshift of finite type in d dimensions with a continuum of ergodic measures of maximal entropy.

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1. Introduction

We consider symbolic dynamical systems whose underlying group structure is $\mathbb{Z}^d$ and whose symbol set is F, a finite set of at least two elements. $\mathbb{Z}^d$ is given the L^1 (nearest neighbor) norm $|\ |$ so that $|(x_1, x_2, \ldots, x_d)| = |x_1| + |x_2| + \cdots + |x_d|$. (We will later consider the L^∞ norm $|\ |_\infty$ given by $|x|_\infty = \max_i |x_i|$, but if no subscript is attached, it is understood to be the L^1 norm.) If A is a subset of $\mathbb{Z}^d$ then the **boundary** of A is $\partial A = \{x \in A: \exists y \in A^c \text{ with } |x - y| = 1\}$.

A **configuration** is a map η: $A \subseteq \mathbb{Z}^d \longrightarrow F$. $x \in A$ are called locations and $\eta(x)$ is the **value** of the configuration at location x. Usually A will be a finite set or $\mathbb{Z}^d$ itself. A configuration η: $A \longrightarrow F$ is a **restriction** of a configuration ζ: $B \longrightarrow F$ if $A \subseteq B$ and ζ agrees with η on A. We also say in this case that ζ is an **extension** of η. Note that $\mathbb{Z}^d$ acts on configurations by translations. If $y \in \mathbb{Z}^d$ set for $x \in \mathbf{Z}^d$, $T_y(x) = x - y$ and for $A \subseteq \mathbf{Z}^d$, set $T_y A = \{x - y: x \in A\}$. If η: $A \longrightarrow F$, we also let $T_y\eta(x) = \eta(T_y(x))$ for $x \in T_{-y}A$.

Definition 1.1: Let η_i: $A_i \longrightarrow F; 1 \le i \le K$ be a finite set S of configurations with A_i finite for each $1 \le i \le K$. The **subshift of finite type** (in d dimensions) corresponding to S is the set $X \subseteq F^{\mathbb{Z}^d}$ consisting of all configurations η: $\mathbb{Z}^d \longrightarrow F$ such that for all $y \in \mathbb{Z}^d$ it is not the case that $T_y\eta$ is an extension of some η_i. (The η_i's should be thought of as the disallowed finite configurations.)

Definition 1.2: An $X \subseteq F^{\mathbb{Z}^d}$ is a **symmetric nearest neighbor system** if there is a subset $G \subseteq F \times F$ that is symmetric (i.e. $(e, f) \in G \Rightarrow (f, e) \in G$) and such that $X = \{\eta: \mathbb{Z}^d \longrightarrow F: x, y \in \mathbb{Z}^d, |x - y| = 1 \text{ implies } (\eta(x), \eta(y)) \notin G\}$. A **nearest neighbor system** is a subshift of finite type where all of the A_i's consist of 2 points x and y with $|x - y| = 1$.

Note that a symmetric nearest neighbor system is of course a nearest neighbor system. If X is a subshift of finite type then X is closed in the usual product topology and is shift invariant, i.e., $\eta \in X$ and $y \in \mathbb{Z}^d$ implies $T_y(\eta) \in X$.

The next definition gives a measure of the size of a subshift of finite type X or its degree of complexity. If $\tilde{\eta}$: $A \longrightarrow F$ is a configuration, we say that $\tilde{\eta}$ is **compatible** with X if $\exists \eta \in X$ such that $\tilde{\eta}$ is a restriction of η. Let $\Lambda_n = [-n, n]^d$ and $X_n = \{\tilde{\eta}: \Lambda_n \longrightarrow F \text{ with } \tilde{\eta} \text{ compatible}\}$. We also let $N_n = |X_n|$ ($|A|$ is the cardinality of A) and $X(\tilde{\eta}) = \{\eta \in X: \eta \text{ is an extension of } \tilde{\eta}\}$.

Definition 1.3: The **topological entropy** of X is

$$H(X) = \lim_{n\to\infty} \frac{\log N_n}{|\Lambda_n|}.$$

Suppose that μ is a translation invariant probability measure on X. Then the **measure theoretic entropy** of μ is

$$H(\mu) = \lim_{n\to\infty} -\frac{1}{|\Lambda_n|} \sum_{\tilde{\eta}\in X_n} \mu(X(\tilde{\eta})) \log \mu(X(\tilde{\eta})).$$

Both of these limits exist by subadditivity. Clearly for any such measure μ we have $H(\mu) \le H(X)$. In fact we have the following variational principle. See [10] for an elementary proof.

THEOREM 1.4: *Let X be a subshift of finite type. Let $\mathcal{M}_X$ be the set of translation invariant measures on X. Then $H(X) = \sup_{\mu\in\mathcal{M}_X} H(\mu)$ and moreover the supremum is achieved at some measure.*

In this paper, we will only consider so called strongly irreducible subshifts of finite type.

Definition 1.5: Let X be a subshift of finite type. X is **strongly irreducible** if there is an $r \ge 0$ such that whenever we have finite configurations $\eta_1\colon A_1 \longrightarrow F$ and $\eta_2\colon A_2 \longrightarrow F$ compatible with X and the distance between A_1 and A_2 is greater than r, there is then an $\eta \in X$ that is an extension of both η_1 and η_2.

This paper is concerned with the question of whether this supremum is achieved at more than one place, that is, whether there is more than one measure of maximal entropy. It has recently been shown in [3] that there are strongly irreducible subshifts of finite type with more than 1 measure of maximal entropy. One of the main points of this paper is to present a number of theorems concerning the uniqueness of the measure of maximal entropy. Before doing this however, we first prove the following two theorems concerning nonuniqueness.

THEOREM 1.6: *For any $k \ge 1$ and $d \ge 2$, there exists a strongly irreducible subshift of finite type in d dimensions which has exactly k ergodic measures of maximal entropy.*

It is natural to ask if there exists a strongly irreducible subshift of finite type which has infinitely many ergodic measures of maximal entropy. The following result provides an answer for $d \ge 3$. We do not know if this is possible for $d = 2$.

THEOREM 1.7: *For any $d \geq 3$, there exists a strongly irreducible subshift of finite type in d dimensions which has a continuum of ergodic measures of maximal entropy.*

In [3], the main subshift of finite type which was analyzed was the following.

Example 1.8: Let M be a positive integer and $F = \{-M, -M+1, \ldots, -2, -1, 1, 2, \ldots, M-1, M\}$. Consider the symmetric nearest neighbor system given by $G = \{(i,j) \in F \times F\colon ij \leq -2\}$. In words, a negative may not sit next to a positive unless they are each ± 1.

We list here the three main results which were proven for Example 1.8 in [3]. To do this, we first need a definition.

Definition 1.9: Given $\eta \in X$, we say $G \subseteq \mathbf{Z}^d$ is a **positive cluster** (with respect to η) if G is connected (using the usual nearest neighbor notion), $\eta(x) \geq 1$ for all $x \in G$ and G is maximal with respect to these two properties. A **negative cluster** is defined analogously. A **cluster** is either a positive or negative cluster.

THEOREM 1.10: *Consider the subshift of finite type given by Example 1.8 and let $d \geq 2$. If*

$$M > 4e28^d,$$

then given any ergodic measure of maximal entropy in d–dimensions, either there is almost surely exactly one infinite positive cluster whose complement contains no infinite connected subset or there is almost surely exactly one infinite negative cluster whose complement contains no infinite connected subset.

Because of the $\pm$ symmetry, Theorem 1.10 immediately yields

COROLLARY 1.11: *Consider the subshift of finite type given by Example 1.8 and let $d \geq 2$. If*

$$M > 4e28^d,$$

then there is more than 1 measure of maximal entropy in d–dimensions.

THEOREM 1.12: *Consider the subshift of finite type given by Example 1.8 and let $d \geq 2$. If*

$$M > 4e28^d,$$

then there are exactly 2 ergodic measures of maximal entropy in d dimensions.

Note that Theorem 1.12 proves Theorem 1.6 for the case $k = 2$. Our first uniqueness result concerns a simple modification of Example 1.8 where the important $\pm$ symmetry no longer exists.

Example 1.13: Let M be a positive integer, $1 \leq k \leq M - 1$ and $F = \{-M + k, \ldots, -2, -1, 1, 2, \ldots, M - 1, M\}$. Consider the symmetric nearest neighbor system given by $G = \{(i,j) \in F \times F : ij \leq -2\}$.

THEOREM 1.14: *Consider the subshift of finite type given in Example 1.13. If $M > 4e(28)^d$ and $1 \leq k \leq M - 1$, then there exists a unique measure of maximal entropy in d–dimensions.*

The above theorem demonstrates the importance of the symmetry present in Example 1.8 in terms of having more than one measure of maximal entropy. Our second uniqueness result in the nonsymmetric case is the following.

THEOREM 1.15: *Consider the subshift of finite type given in Example 1.13. If*

$$\frac{M-k}{M} < \frac{1}{(2d-1)2^{2d+1}},$$

then there exists a unique measure of maximal entropy in d–dimensions.

The above two results are special cases of the following conjecture which we believe is true.

CONJECTURE 1.16: *The subshift of finite type given in Example 1.13 always has a unique measure of maximal entropy.*

Our next result tells us that if we take any nearest neighbor system (not necessarily strongly irreducible) and add enough new states all of which can be adjacent to any other state including each other, we obtain a strongly irreducible subshift of finite type with a unique measure of maximal entropy. To state this result, we need to introduce percolation.

Suppose each vertex $i \in \mathbf{Z}^d$ is, independent of all other vertices, **open** with probability p_i and **closed** with probability $1 - p_i$. Denote the corresponding probability measure by $P_{\{p_i\}}$. For a realization of the process a path is called open if all its vertices are open. We say that **percolation** occurs if $P_{\{p_i\}}$(there exists an infinite open path) > 0 (in which case this probability is 1 since the event is a tail event). If all the p_i's are equal, say p, we write P_p for the above probability measure and define the **critical probability** $p_c(d) =$

$\inf\{p: P_p(\text{there exists an infinite open path}) > 0\}$. One of the first results in percolation was to show that $p_c(2) < 1$ ([2]). (It is easy to show that for all d, $p_c(d) > 0$ which will tell us that Theorem 1.17 is nonvacuous.) The above model is called **independent site percolation**. For further study, see [6] and [9].

THEOREM 1.17: *Given a nearest neighbor system X in d dimensions, let X^n denote the subshift of finite type obtained by adding n new states all of which can be adjacent to any other state including each other. If the number of states for X is k and*

$$\frac{2k}{n+k} - \left(\frac{k}{n+k}\right)^2 < p_c(d),$$

then X^n has a unique measure of maximal entropy.

Theorem 1.17 together with the methods of [3] allows us to prove a phase transition in a particular symmetric nearest neighbor system which we call the "iceberg" model. The state space has positives, negatives and zeroes. More precisely, the states are

$$\{-M, \ldots, -1, (0,1), (0,2), (0,3), (0,4), (0,5), 1, \ldots, M\}.$$

We think of this as having five zeroes. The rules are that the positives may not sit next to the negatives but all else is allowed (and so in particular the five 0's can sit next to anyone).

THEOREM 1.18: *Consider the "iceberg" model for d=2. Then for $M = 1$, there is a unique measure of maximal entropy while for large M, there is more than one measure of maximal entropy.*

We call this a phase transition since depending on the parameter M, it is possible to have either one or more measures of maximal entropy. The advantage of this model over Example 1.8 is that we haven't been able to show that there is a unique measure of maximal entropy for small M (although we certainly believe this) demonstrating the desired phase transition.

We find it convenient to mention one other result from [3] which we will need here. We state a stronger version than stated there but which can be proved in the same manner. We point out that it is exactly this theorem that allows one to begin an analysis of measures of maximal entropy. We first give a definition. Letting r be a positive integer and $G \subseteq \mathbf{Z}^d$ be finite, we define $B^r = B^r(G)$ to be

$$\{x \notin G: \exists y \in G \text{ with } |x-y| \leq r\}.$$

Definition 1.19: A measure μ on $S^{\mathbf{Z}^d}$ is r–**Markov** if for any finite $G \subseteq \mathbf{Z}^d$, the conditional distribution of μ on G given the values on G^c is the same as the conditional distribution of μ on G given the values on $B^r(G)$.

PROPOSITION 1.20: *Let μ be a measure of maximal entropy for a subshift of finite type where for each A_i (defining the subshift of finite type), we have that $\sup_{x,y\in A_i}|x-y| \le r$. Then μ is r–Markov and furthermore the conditional distribution of μ on any finite set G given the configuration on $B^r(G)$ is μ–a.s. uniform over all configurations on G which (together with the configuration on $B^r(G)$) extend the configuration on $B^r(G)$.*

We mention that Proposition 1.20 is proved in [3] in the case where the subshift of finite type is a symmetric nearest neighbor system (Proposition 1.19 in that paper) while the more general result can be proved in the same way. Note that for nearest neighbor systems, the measures of maximal entropy are then 1–Markov which becomes the usual definition of Markov. In this paper, we will only apply Proposition 1.20 in the nearest neighbor case.

There is a converse to Proposition 1.20 which is much easier to prove but which only necessarily holds in the strongly irreducible case. The proof given in [3] for the symmetric nearest neighbor case can easily be carried out in this more general setting.

PROPOSITION 1.21: *Let μ be a measure defined on a strongly irreducible subshift of finite type X where for each A_i, we have that $\sup_{x,y\in A_i}|x-y| \le r$. Assume that μ is r–Markov and furthermore that the conditional distribution of μ on any finite set G given the configuration on $B^r(G)$ is μ–a.s. uniform over all configurations on G which (together with the configuration on $B^r(G)$) extend the configuration on $B^r(G)$. Then μ has maximal entropy.*

The rest of this paper is devoted to the proofs of the above results.

2. A subshift with exactly k ergodic measures of maximal entropy in $d \geq 2$ dimensions

In this section, we construct a subshift of finite type with exactly k ergodic measures of maximal entropy in d–dimensions, thereby proving Theorem 1.6. This system will be a natural generalization of Example 1.8. We first introduce the following notions which will be used throughout this paper.

Definition 2.1: $G \subseteq \mathbf{Z}^d$ is **connected** if for each pair $x, y \in G$, there is a path $x = x_0, x_1, \ldots, x_n = y$ with $x_i \in G$, for $0 \leq i \leq n$ and $|x_i - x_{i-1}| = 1$, for $1 \leq i \leq n$.
$G \subseteq \mathbf{Z}^d$ is **$*$–connected** if for each pair $x, y \in G$, there is a path as above with $|x_i - x_{i-1}|_\infty = 1$ for $1 \leq i \leq n$.
$G \subseteq \mathbf{Z}^d$ is **$**$–connected** if for each pair $x, y \in G$, there is a path as above with $|x_i - x_{i-1}|_\infty \leq 3$ for $1 \leq i \leq n$.

Paths of the second and third type will be called respectively $*$–paths and $**$–paths while paths of the first type will simply be called paths. We will assume all paths in this paper are nonintersecting.

The system we construct is given as follows. We let $F = \{(i,j): 1 \leq i \leq k, 1 \leq j \leq M\}$ where k and M are two positive integer parameters. Consider the symmetric nearest neighbor system X given by $G = \{((i,j),(i',j') \in F \times F: i \neq i'$ and at least one of j and j' is not $1\}$. This is defined for any dimension d.

Pictorially, we have k values $\{(i,1): 1 \leq i \leq k\}$ all of which can sit next to each other. Each of these values has in turn $M-1$ values which only it can sit next to. Namely, $(i,1)$ can sit next to $\{(i,j): 2 \leq j \leq M\}$. Moreover, these $M-1$ values can all sit next to each other. It is clear that this subshift is strongly irreducible and moreover that when $k=2$, we recover Example 1.8. Throughout this section, we will consider only this subshift which has of course the parameters M, k and d where d is the dimension of the lattice. We will show that if M is sufficiently large relative to k and d (the precise requirement being given later), then there are exactly k ergodic measures of maximal entropy. We mention that the proof that there are at least k ergodic measures of maximal entropy follows more or less the proof for $k=2$ given in [3]. The fact that there are exactly k however requires a different method.

For $\ell = 1, \ldots, k$, we let

$$C_\ell = \{(\ell, j): 1 \leq j \leq M\} \quad \text{and} \quad C'_\ell = C_\ell \backslash \{(\ell, 1)\}.$$

Definition 2.2: Given $\eta \in X$, we say $G \subseteq \mathbf{Z}^d$ is an **ℓ–cluster** (with respect to η) if G is connected (using the usual nearest neighbor notion), $\eta(x) \in C_\ell$ for all $x \in G$ and G is maximal with respect to these properties.

THEOREM 2.3: *Consider the subshift of finite type given above and let $d \geq 2$. If*

$$M > 2ek(7k^2)^d,$$

then given any ergodic measure of maximal entropy, there is an $\ell \in \{1, \ldots, k\}$ such that there is almost surely exactly one infinite ℓ–cluster whose complement contains no infinite connected subset.

Note that Theorem 2.3 immediately gives because of symmetry the following corollary.

COROLLARY 2.4: *Consider the subshift of finite type given above and let $d \geq 2$. If*

$$M > 2ek(7k^2)^d,$$

then there are at least k ergodic measures of maximal entropy.

The proof of Theorem 2.3 rests on the following lemma. Once this lemma is proved, we will prove Theorem 2.3 by using the same method used to prove Theorem 1.15 in [3] (which is Theorem 1.10 in the present paper). We first introduce the following notation which will be used throughout this paper.

Definition 2.5: Let $S \subseteq \mathbf{Z}^d$ be finite. Let η be a compatible configuration defined on ∂S. Define Ω_η to be the set of compatible configurations on S which extend η. Define μ_η to be uniform measure on Ω_η.

LEMMA 2.6: *Let $B = \{(i, 1)\colon 1 \leq i \leq k\}$. With M, k and d fixed, let μ be a measure of maximal entropy on X. Let $G \subseteq \mathbf{Z}^d$ and E be the event*

$$\{\eta(x) \in B \text{ for all } x \in G\}.$$

Then $\mu(E) \leq \left(\frac{k^{2d+1}}{M}\right)^{|G|}$

Proof: Call a path in $\mathbf{Z}^d$ **special** (with respect to η) if no two successive points on the path are in B. Let

$$A(x) = \{y\colon \text{ there exists a special path from } x \to y\}.$$

The existence of a special path between x and y is an equivalence relation whose equivalence classes we call **atoms**. $A(x)$ above is therefore simply the atom containing x.

Before continuing with the proof, we give another construction of an atom which will hopefully allow the reader to better understand this proof. We say that $A \subseteq \mathbf{Z}^d$ is **flippable** (for η) if for any $\ell = 1, \ldots, k$, if we change the configuration

η on A by setting all the first coordinates of the values to be ℓ, then we still have an element of X. The reader should check that if this is true for at least two values of ℓ, then A is flippable and that an atom is exactly a minimal flippable set. The importance of flippable sets and atoms is that when any subset of the flippable sets are flipped, the collection of flippable sets for the new configuration and hence the collection of atoms is unchanged. This is what makes the proof work.

Now, consider the mapping T: $X \to X$ as follows. Consider $\eta \in X$ and let $\eta(x) = (i,j)$. If $|A(x)| < \infty$, let $T(\eta)(x) = \eta(x)$. If $|A(x)| = \infty$, let $T(\eta)(x) = (k,j)$. Let $\tilde{\mu}$ be the measure on X obtained from μ and T. The proof of Proposition 4.5 in [3] can easily be modified to give us that μ is also a measure of maximal entropy. (Essentially, the reason that the transformed measure has maximal entropy is that when we flip all the infinite atoms to be of one type, we are only giving up entropy 0 information since the number of infinite atoms in a box of size n^d is at most cn^{d-1}.) Moreover, clearly $\mu(E) = \tilde{\mu}(E)$. Let $\alpha = \tilde{\mu}(E)$. We want to show $\alpha \leq \left(\frac{k^{2d+1}}{M}\right)^{|G|}$. If $\alpha = 0$, there is nothing to prove and so we assume $\alpha > 0$ and let $\epsilon < \alpha$. By definition of $\tilde{\mu}$, we can take Λ_n containing $G \cup \partial G^c$ with n so large that with probability greater than $1-\epsilon$, any atom with values in $C_\ell, \ell \neq k$ (which is necessarily finite) intersecting either G or ∂G^c is contained in Λ_{n-1}. Letting F be this event, we have that $\tilde{\mu}(F) > 1-\epsilon$ and so $\tilde{\mu}(E \cap F) > \alpha - \epsilon$. There is then a configuration η with location set $\partial\Lambda_n$ such that $\tilde{\mu}(\eta) > 0$ and $\tilde{\mu}(|\eta \text{ on } \partial\Lambda_n)$ gives $E \cap F$ probability $> \alpha - \epsilon$. Proposition 1.20 tells us that this conditional measure restricted to Λ_n is uniform distribution μ_η on Ω_η (see Definition 2.5). Note that the events E and F are measurable with respect to the configuration on Λ_n and so we can think of them as subsets of Ω_η.

Next, for each configuration $\xi \in \Omega_\eta \cap E \cap F$, let $\tilde{\xi}$ be the configuration obtained from ξ as follows. If $x \in G \cup \partial G^c$ and $\xi(x) \notin C_k$, then for all $y \in A(x)$ (necessarily contained in Λ_{n-1}), let $\tilde{\xi}(y) = (k,j)$ if $\xi(y) = (i,j)$. ξ is unchanged on all other points. Note that if $x \in G \cup \partial G^c$ with $\xi(x) \in C_k$, then $\tilde{\xi} = \xi$ on $A(x)$. In words, we are taking the atoms which intersect $G \cup \partial G^c$ and making all the first components have value k. Note also that $\tilde{\xi} \in \Omega_\eta \cap E \cap F$ and $\tilde{\xi}(x) = (k,1)$ for each $x \in G$. Since

$$|G \cup \partial G^c| \leq (2d+1)|G|,$$

the set $\{\xi' \in \Omega_\eta \cap E \cap F | \tilde{\xi}' = \tilde{\xi}\}$ has cardinality at most $k^{(2d+1)|G|}$. This is because $\tilde{\xi}$ and ξ have the same atoms and hence from $\tilde{\xi}$, one can recover ξ up to which set C_ℓ the atoms touching $G \cup \partial G^c$ belong to.

Given $\xi \in \Omega_\eta \cap E \cap F$, define the class $C(\tilde{\xi})$ to be

$$\{\rho | \rho \text{ is a configuration on } \Lambda_n \\ \text{with } \rho(x) = \tilde{\xi}(x) \text{ for } x \notin G \text{ and } \rho(x) \in C_k \text{ for } x \in G\}.$$

Clearly $C(\tilde{\xi}) \subseteq \Omega_\eta \cap F$, $|C(\tilde{\xi})| = M^{|G|}$ and $C(\tilde{\xi}) \cap C(\tilde{\xi}') = \emptyset$ for $\tilde{\xi} \neq \tilde{\xi}'$ since if $\tilde{\xi} \neq \tilde{\xi}'$, they must differ at some point not in G as they are $\equiv (k, 1)$ on G.

We therefore obtain

$$\begin{aligned} \alpha - \epsilon < \mu_\eta(E \cap F) &= \frac{|\Omega_\eta \cap E \cap F|}{|\Omega_\eta|} \\ &\leq \frac{|\{\tilde{\xi} | \xi \in \Omega_\eta \cap E \cap F\}| k^{(2d+1)|G|}}{|\{\tilde{\xi} | \xi \in \Omega_\eta \cap E \cap F\}| M^{|G|}} = \left(\frac{k^{2d+1}}{M}\right)^{|G|}. \end{aligned}$$

As ϵ is arbitrary, we obtain the desired inequality. ∎

Before proceeding with the proof of Theorem 2.3, we need the following. Lemma 2.8 below is proved in [3].

Definition 2.7: $G \subseteq \mathbf{Z}^d$ is an **enclosing** set if

(i) $0 \in G$,

(ii) G is finite,

(iii) G and G^c are each connected.

If $H \subseteq \mathbf{Z}^d$ is any finite set that contains 0 and is connected, then H^c has a unique infinite component and (possibly) some finite number of finite components. Note that $H \cup \{x | \, x \in$ finite component of $H^c\}$ is enclosing.

LEMMA 2.8: *The number of enclosing sets G with $\ell = |\partial G|$ is $\leq \binom{\ell+2d-4}{2d-2} (e7^d)^\ell$.*

Proof of Theorem 2.3: For this proof, we will simply follow the proof of Theorem 1.15 in [3] (which is Theorem 1.10 in the present paper).

Let μ be an ergodic measure of maximal entropy. With B defined as in Lemma 2.6, if G is an enclosing set, let

$$E_G = \{\eta | \eta(x) \in B \; \forall \, x \in \partial G\}.$$

Note that Lemma 2.6 immediately gives us that $\mu(E_G) \leq \left(\frac{k^{2d+1}}{M}\right)^{|\partial G|}$. We also let $F = \bigcup_{G:\, G \text{ is enclosing}} E_G$.

It is easy to see that F^c implies the existence of an infinite ℓ–cluster containing the origin for some ℓ. The first step is to show that $\mu(F) < 1$. We have using Lemma 2.8

$$\mu(F) \leq \sum_{G:\, G \text{ is enclosing}} \mu(E_G) \leq \sum_{\ell=1}^{\infty} \binom{\ell+2d-4}{2d-2} (e7^d)^\ell \left(\frac{k^{2d+1}}{M}\right)^\ell .$$

Finally a simple computation gives

$$\sum_{\ell=1}^{\infty} \binom{\ell+2d-4}{2d-2} (1/2)^\ell = 1.$$

Hence if $M > 2(e7^d k^{2d+1}) = 2ek(7k^2)^d$, the above sum is < 1, as desired.

For any ℓ, having an infinite ℓ–cluster has probability 0 or 1 by ergodicity. Therefore, since $\mu(F) < 1$, we know that for some ℓ, there is an infinite ℓ–cluster a.s..

To prove the rest of Theorem 2.3, we need to show that if C is an infinite ℓ–cluster for some ℓ, then $\mathbf{Z}^d \cap C^c$ has all finite components a.s. First, it can be shown that if C is an infinite ℓ–cluster for some ℓ and $\mathbf{Z}^d \cap C^c$ has some infinite component, then there must be an infinite $**$–path with every value on this path being in B. (This geometric argument is given in [3] and a more complicated version will be given in Theorem 2.11 below.) The argument is finally completed by showing that the probability that there is an infinite $**$–path with every value on this path being in B is 0. This is easily done by using Lemma 2.6 together with a trivial upper bound on the number of $**$–paths of length n starting from the origin. ∎

We now know that there are at least k ergodic measures of maximal entropy. We want to finally prove that there are exactly k ergodic measures of maximal entropy. Theorem 1.12 in the present paper which was proven in [3] shows that this is the case when $k = 2$. Unfortunately, $k = 2$ is a special case since there is then a natural partial order that one can put on F which is compatible in some sense with the subshift of finite type. When $k \geq 3$, there is no such partial order and so a different method is required. We first need the following proposition which is an easy generalization of the main result in [1]. This result is valid for general specifications which we do not define here.

LEMMA 2.9 [1]: *Let X be a nearest neighbor symmetric subshift of finite type with values F (not necessarily strongly irreducible). Assume that there exists*

$P \subseteq F$ such that for any two elements x, y in P, $(x, y) \notin G$ where G defines the nearest neighbor symmetric subshift (this means that x and y are allowed to sit next to each other) and such that given $x, y \in P$ and $z \notin P$, $(x, z) \in G$ if and only if $(y, z) \in G$. (In words, all the elements of P behave in the same way.) Let μ and ν be two measures of maximal entropy for X and let $E \subseteq X \times X$ be the set of pairs of configurations (η, δ) such there exists an infinite path π in $\mathbf{Z}^d$ such that for all $x \in \pi$, at least one of $\eta(x)$ and $\delta(x)$ is not in P. If

$$\mu \times \nu(E) = 0,$$

then $\mu = \nu$.

To be able to apply Lemma 2.9, we need a stronger version of Theorem 2.3.

Definition 2.10: Given $(\eta, \delta) \in X \times X$, we say $G \subseteq \mathbf{Z}^d$ is an (ℓ_1, ℓ_2)–**double–cluster** (with respect to (η, δ)) if G is connected (using the usual nearest neighbor notion), $(\eta(x), \delta(x)) \in C'_{\ell_1} \times C'_{\ell_2}$ for all $x \in G$ and G is maximal with respect to these properties.

THEOREM 2.11: *Let $d \geq 2$ and $M > 8^{2(2d+1)} e^{2(2d-1)} 7^{2d(2d-1)} 4k^{2d+1}$. Let μ and ν be two ergodic measures of maximal entropy. Assume that there exists an infinite ℓ–cluster whose complement contains no infinite component a.s. with respect to both μ and ν. (Theorem 2.3 tells us both μ and ν have such an infinite cluster for some ℓ, but the point is that we assume that μ and ν have the same type cluster.) Then there is $\mu \times \nu$–almost surely exactly one infinite (ℓ, ℓ)–double–cluster whose complement contains no infinite connected subset.*

Before giving the proof of Theorem 2.11, we show how this completes the proof that there are exactly k ergodic measures of maximal entropy.

THEOREM 2.12: *Consider the subshift of finite type given above and let $d \geq 2$. If*

$$M > 8^{2(2d+1)} e^{2(2d-1)} 7^{2d(2d-1)} 4k^{2d+1},$$

then there are exactly k measures of maximal entropy.

Proof: By Corollary 2.4 above, there are at least k ergodic measures of maximal entropy. By Theorem 2.3 above, it suffices to show that if $\ell \in \{1, \ldots, k\}$, and μ and ν are two ergodic measures of maximal entropy such that for each of them, there is a.s. an infinite ℓ–cluster whose complement contains no infinite component, then $\mu = \nu$.

In view of Lemma 2.9 above, it suffices to show that

$$\mu \times \nu(E) = 0$$

where E is the event that there exists an infinite path π: $0 \to \infty$ such that for all $x \in \pi$, at least one of $\eta(x)$ and $\delta(x)$ is not in C'_ℓ. However, this result is part of the content of Theorem 2.11. ∎

The rest of this section is devoted to the proof of Theorem 2.11.

Definition 2.13: Let G be enclosing. Define the event $E_G \subsetneq X \times X$ to be

$$\{(\eta,\delta)| \bigcup_{x\in\partial G} \{\eta(x),\delta(x)\} \cap B = \emptyset \text{ and for all } x \in \partial G^c, \eta(x) \text{ or } \delta(x) \text{ is in } B\}.$$

We also let

$$E_\emptyset = \{\eta(0) \in B\} \cup \{\delta(0) \in B\} \text{ and}$$

$$F = E_\emptyset \cup \bigcup_{G:\, G \text{ is enclosing}} E_G.$$

It is easy to see that F^c implies the existence of an infinite (ℓ_1, ℓ_2)–double–cluster for some ℓ_1 and ℓ_2.

We can show that $\mu \times \nu(E) > 0$ where E is the event that there exists an infinite (ℓ_1, ℓ_2)–double–cluster for some ℓ_1 and ℓ_2. Unfortunately, we would not necessarily be able to conclude that E has probability 1 since we don't know that $\mu \times \nu$ is ergodic even though both μ and ν are. We therefore need to consider the ergodic decomposion of $\mu \times \nu$. Since μ and ν are ergodic, almost all elements that arise in the ergodic decomposition have μ and ν as their respective marginals.

LEMMA 2.14: *Let m be an ergodic measure on $X \times X$ which has μ and ν as its two marginals where d, M, μ and ν are as in Theorem 2.11. Then $m(E) = 1$ where E is the event that there exists an infinite (ℓ, ℓ)–double–cluster.*

Proof: It suffices to show that there is an infinite (ℓ_1, ℓ_2)–double–cluster for some ℓ_1 and ℓ_2 with probability 1 since by Theorem 2.3 there clearly exists an (ℓ_1, ℓ_2)–double–cluster for $(\ell_1, \ell_2) \neq (\ell, \ell)$ with m–probability 0. By ergodicity, it suffices to show that there is positive probability of having an infinite (ℓ_1, ℓ_2)–double–cluster for some ℓ_1 and ℓ_2. Since it is obvious that F^c implies the existence of an infinite (ℓ_1, ℓ_2)–double–cluster containing the origin for some ℓ_1 and ℓ_2, we need only show that $m(F) < 1$.

It follows by definition that if E_G occurs (G being an enclosing set), then for all $x \in \partial G^c$, at least one of $\eta(x), \delta(x)$ is in B. Hence one of η and δ has at least $|\partial G^c|/2$ points of ∂G^c where the value is in B. Since there are at most $2^{|\partial G^c|}$ such subsets, Lemma 2.6 easily gives us that

$$m(E_G) \leq 2(2^{|\partial G^c|}) \left(\frac{k^{2d+1}}{M}\right)^{\frac{|\partial G^c|}{2}}.$$

(Note that the values for η and δ need not be independent.)

Using the easy bound $|\partial G^c| \geq \frac{|\partial G|}{2d-1}$ together with the fact that $M > 4k^{2d+1}$, one obtains

$$m(E_G) \leq 2 \left(\frac{2^{\frac{1}{2d-1}} k^{\frac{2d+1}{2(2d-1)}}}{M^{\frac{1}{2(2d-1)}}}\right)^{|\partial G|}.$$

Since $E_\emptyset$ clearly has measure at most $2\frac{k^{2d+1}}{M}$ by Lemma 2.6, Lemma 2.8 and the above give us

$$\begin{aligned} m(F) &\leq m(E_\emptyset) + \sum_{G:\, G \text{ is enclosing}} m(E_G) \\ &\leq 2\frac{k^{2d+1}}{M} + \sum_{\ell=1}^{\infty} \binom{\ell + 2d - 4}{2d-2} 2 \left(\frac{e7^d 2^{\frac{1}{2d-1}} k^{\frac{2d+1}{2(2d-1)}}}{M^{\frac{1}{2(2d-1)}}}\right)^{\ell}. \end{aligned}$$

Next, a simple computation gives

$$\sum_{\ell=1}^{\infty} \binom{\ell + 2d - 4}{2d-2} (1/2)^{\ell} = 1.$$

An easy calculation then shows that if $M > 8^{2(2d+1)} e^{2(2d-1)} 7^{2d(2d-1)} 4k^{2d+1}$, then $m(F) < 1$, as desired. ∎

PROPOSITION 2.15: *Let d, M, μ and ν be as in Theorem 2.11. Then $\mu \times \nu(E) = 1$ where E is the event that there exists an infinite (ℓ, ℓ)–double–cluster.*

Proof: Consider the ergodic decomposition of $\mu \times \nu$. The ergodicity of μ and ν imply that almost every element in the ergodic decomposition of $\mu \times \nu$ has marginals μ and ν respectively. By Lemma 2.14, almost every element of the ergodic decomposition therefore gives E probability 1 and therefore $\mu \times \nu$ must also. ∎

LEMMA 2.16: *If*

$$M > 2ek(7k^2)^d,$$

then given any two measures μ *and* ν *of maximal entropy, the* $\mu \times \nu$*-probability of having an infinite* $**$*-path* π *such that for all* $x \in \pi$*, at least one of* $\eta(x)$ *and* $\delta(x)$ *is in* B *is 0.*

Proof: It suffices to show that there is no such path containing the origin 0 a.s. Given a fixed $**$-path γ of length n, the probability that for all $x \in \gamma$, at least one of $\eta(x)$ and $\delta(x)$ is in B is by Lemma 2.6 at most $2^{n+1}\left(\frac{k^{2d+1}}{M}\right)^{n+1}$ since η must be in B on some subset of the path, δ must be in B on the complement of this subset, and there are at most 2^{n+1} subsets.

As the number of $**$-paths of length n is at most $(7^d)^n$, the probability that there exists a $**$-path of length n starting at 0 of the above form is at most $(7^d 2k^{2d+1}/M)^{n+1}$ which $\to 0$ as $n \to \infty$ as $M > 2ek(7k^2)^d$. ∎

The next lemma is proven in [3].

LEMMA 2.17: *Let* G *be an enclosing set. Then* ∂G *is* $**$*-connected.*

Proof of Theorem 2.11: By Proposition 2.15, there is some infinite (ℓ, ℓ)-double-cluster C a.s.. We want to show that $\mathbf{Z}^d \cap C^c$ has all finite components a.s..

In view of Proposition 2.16, we need to prove the deterministic fact that if $(\eta, \delta) \in X \times X$ is such that there is an infinite (ℓ, ℓ)-double-cluster C and some component of $\mathbf{Z}^d \cap C^c$ is infinite, then there is an infinite $**$-path π such that for all $x \in \pi$, at least one of $\eta(x)$ and $\delta(x)$ is in B. By compactness, it suffices to show that there is some point from which there begins a $**$-path of any length k such that at least one of $\eta(x)$ and $\delta(x)$ is in B for all points on this path.

We therefore let I be an infinite component of C^c. Clearly there must be some point in I which is adjacent to some point in C which we assume without loss of generality is 0. Note that each point of ∂I must necessarily have either $\eta(x)$ or $\delta(x)$ in B. Let I' be the connected component of $I \cap \Lambda_n$ which contains 0. It is clear that I' has a connected complement and hence I' is an enclosing set.

By Lemma 2.17, $\partial I'$ is $**$-connected. Clearly 0 and any point of $I' \cap \partial\Lambda_n$ (which clearly is nonempty as I is infinite) is in $\partial I'$. Hence there is a $**$-path $\gamma \subseteq \partial I'$ from 0 to some point on $\partial\Lambda_n$. Note next that if $x \in \partial I' \cap \Lambda_{n-1}$, then at least one of $\eta(x)$ and $\delta(x)$ is in B since $\partial I' \cap \Lambda_{n-1} \subseteq \partial I$. (Note that this need not be true for those x in $\partial I' \cap \partial \Lambda_n$.) As n is arbitrary, we obtain $**$–paths of the desired type starting from 0 of any given length k. ∎

3. A subshift with a continuum of ergodic measures of maximal entropy in $d \geq 3$ dimensions

In this section, we prove Theorem 1.7. We carry out the argument only for $d = 3$, the generalization to higher dimensions being obvious. However, we do not know how to make such a construction for $d = 2$.

Let M be a positive integer, $F = \{-M, \ldots, -2, -1, 1, 2, \ldots, M\}$ and $d = 3$. Our subshift of finite type is defined by requiring that for all integers x, y, x', y' and z with $|x - x'| + |y - y'| = 1$,

$$\eta(x, y, z)\eta(x', y', z) \geq -1.$$

It is easy to see that on each horizontal plane

$$P_z = \{(x, y, z) \colon (x, y) \in \mathbf{Z}^2\},$$

the rules of the subshift are those of Example 1.8 while there are no restrictions in the vertical direction; i.e.,

$$\eta(x, y, z)\eta(x, y, z + 1)$$

is allowed to be anything. While this example is nearest neighbor and symmetric in a certain sense, one should note that it is not a symmetric nearest neighbor system in the sense of Definition 1.2. In particular, it is not isotropic. We let Y denote this subshift of finite type and X denote the subshift of finite type of Example 1.8 with $d = 2$. Note that $Y = X^{\mathbf{Z}}$.

Let $M > 4e28^d$ and let μ^+ and μ^- denote the two distinct ergodic measures of maximal entropy for X (see Theorem 1.12). Let ν_p be product measure on $\{-, +\}^{\mathbf{Z}}$ with density p;

$$\nu_p = \prod_{i \in \mathbf{Z}} (p\delta_+ + (1 - p)\delta_-).$$

Given $\eta \in \{-, +\}^{\mathbf{Z}}$, let μ_η be the measure on Y given by

$$\prod_{i \in \mathbf{Z}} \mu^{\eta(i)}.$$

In words, under μ_η, all different horizontal planes are independent with the ith plane being given the measure μ^+ or μ^- depending on $\eta(i)$. Finally, let

$$\mu_p = \int \mu_\eta d\nu_p(\eta).$$

In words, picking a random configuration according to μ_p is done by picking a configuration $\eta \in \{-,+\}^{\mathbf{Z}}$ according to ν_p and then picking a configuration from Y according to μ_η. The following result gives us Theorem 1.7.

THEOREM 3.1: *If $M > 4e28^d$, then each μ_p for $p \in [0,1]$ is an ergodic measure of maximal entropy and the μ_p's are distinct.*

Proof: The fact that each μ_p is $\mathbf{Z}^3$–invariant is obvious. Letting $a = \mu^+(\eta(0,0) \geq 1)$, we have $\mu_p(\eta(0,0,0) \geq 1) = pa + (1-p)(1-a) = p(2a-1) + (1-a)$. Since $\mu^+ \neq \mu^-$, $a > 1/2$ which gives $2a - 1 > 0$ and we see that the μ_p's are distinct. (Actually to conclude that $a > 1/2$ from the fact that $\mu^+ \neq \mu^-$, one also needs the fact that $\mu^- \preceq \mu^+$ in the sense of the partial order discussed in [3] which is also given in the next section.)

It is clear that each μ_p is Bernoulli (in fact, independent) under the $\mathbf{Z}$–action given by vertical translation and hence each μ_p is ergodic under the full $\mathbf{Z}^3$–action.

We finally show that they all have maximal entropy. Letting a_n denote the number of compatible configurations for X in $[-n,n]^2$, the number of compatible configurations for Y in $[-n,n]^3$ is a_n^{2n+1}. Since $\frac{\ln(a_n^{2n+1})}{(2n+1)^3} \to H(X)$, we see that $H(Y) = H(X)$. We now show that for each p, $H(\mu_p) = H(X)$, demonstrating maximal entropy for the μ_p's.

For any measure m on Y, not necessarily translation invariant, let $H_n(m)$ denote the entropy in the box $[-n,n]^3$ for m. Note that for each η, $H_n(\mu_\eta) = (2n+1)H_n(+)$ where $H_n(+)$ denotes the entropy in $[-n,n]^2$ for μ^+.

Using the usual concavity property of entropy, we obtain

$$H_n(\mu_p) \geq \int_{\{-,+\}^{\mathbf{Z}}} H_n(\mu_\eta) d\nu_p(\eta) = (2n+1)H_n(+).$$

Dividing by $(2n+1)^3$, letting $n \to \infty$ and using the fact that μ^+ has maximal entropy for X, we obtain $H(\mu_p) = H(X)$, as desired. ∎

We mention that while we obtained a strongly irreducible subshift of finite type with a continuum of ergodic measures of maximal entropy, the example is slightly unsatisfying for the following reason. It is easy to check that for $p \neq 0, 1$,

μ_p is not ergodic under the $\mathbf{Z}^2$–action of translations in the x and y directions. It would be more satisfying to obtain an example where there was a continuum of measures all of which were K or mixing or at least totally ergodic (i.e., ergodic under all nonzero subgroups).

4. Uniqueness in the nonsymmetric case

In this section, we consider only Example 1.13 and prove Theorems 1.14 and 1.15.

In the proof of Theorem 1.14, we only consider the case $k = 1$, the other cases being proved in the same way. We first need two results. The first result is analogous to Lemma 2.6 (in the case $k = 2$) and can be proved in the same way, the only formal difference in the statement being that the subshift of finite type X is now different since $-M$ is no longer a possibility. Lemma 4.2 follows easily (by taking a trivial upper bound on the number of $**$–paths of length n starting from the origin) from Lemma 4.1. We therefore skip the proof of these two lemmas.

LEMMA 4.1: *With M and d fixed, let μ be a measure of maximal entropy on X. Let $G \subseteq \mathbf{Z}^d$. Let E be the event*

$$\{\eta(x) = \pm 1 \textit{ for all } x \in G\}.$$

Then $\mu(E) \leq \left(\frac{2^{2d+1}}{M}\right)^{|G|}$.

LEMMA 4.2: *If*

$$M > 4e(28)^d,$$

then given any measure μ of maximal entropy, the probability of having an infinite $**$–path of ± 1's is 0.

We need the following definitions which are essentially taken from [3] and are completely analogous to the definitions given in §2.

Definition 4.3: If $\alpha \in X$, we call a path π **special** (relative to α) if there are no two points in a row on π where α has absolute value 1. ∎

It is easy to see that the existence of a special path between two points in $\mathbf{Z}^d$ defines an equivalence relation on $\mathbf{Z}^d$ (which depends on the configuration α).

Definition 4.4: An **atom** is an equivalence class with respect to the above equivalence relation.

PROPOSITION 4.5: *If $d \geq 2$ and*

$$M > 4e28^d,$$

then given any ergodic measure of maximal entropy in d–dimensions, either there is almost surely exactly one infinite positive atom whose complement contains no infinite connected subset or there is almost surely exactly one infinite negative atom whose complement contains no infinite connected subset.

Proof: The exact same argument used to prove Theorem 2.3 can be used here. One simply uses Lemmas 4.1 and 4.2 above in the appropriate places. Note that this result differs from the case $k = 2$ in two ways, namely that "cluster" has been replaced by "atom" and that $\{-M, \ldots, -1, 1, \ldots, M\}$ has been replaced by $\{-M+1, \ldots, -1, 1, \ldots, M\}$. The latter difference causes no problem at all and for the first difference, one simply observes that F^c (defined in the proof of Theorem 2.3) also implies the existence of an infinite atom containing the origin. ∎

Proof of Theorem 1.14: One first notes that Proposition 4.5 immediately implies that there exists an ergodic measure μ of maximal entropy such that there is almost surely exactly one infinite positive atom whose complement contains no infinite connected subset. To see this, note that if there were not such a measure, then, by Proposition 4.5, there would be an ergodic measure of maximal entropy ν such that there is almost surely exactly one infinite negative atom whose complement contains no infinite connected subset. One could then obtain a measure with one positive atom whose complement contains no infinite connected subset simply by transforming ν by flipping the infinite negative atom. As explained in §2, the atoms of a configuration are unchanged when we flip some of the negative atoms and therefore this mapping would be invertible which implies that the transformed measure also has maximal entropy (and is of course also ergodic).

Next, the proof of Theorem 1.17 in [3] (Theorem 1.12 in the present paper), also shows here that there is only one ergodic measure of maximal entropy with the property that there is almost surely exactly one infinite positive atom whose complement contains no infinite connected subset. Let μ denote this ergodic measure of maximal entropy. We give an alternative description of μ which will be useful for us. To do this, we first let μ_n^+ be the measure on X which gives probability 1 of having an M at all points of $\Lambda_n^c \cup \partial\Lambda_n$ and gives uniform measure on all compatible configurations on Λ_n which equal M on $\partial\Lambda_n$. It is shown in [3] that μ_n^+ converges as $n \to \infty$ and moreover the discussion there shows that this limit is μ above. (See the proof of Theorem 1.15 below for more details concerning this point.)

From the above discussion, in order to show that there is a unique measure of maximal entropy, it suffices to show that there is no ergodic measure of maximal entropy ν such that there is almost surely exactly one infinite negative atom whose complement contains no infinite connected subset. If such a ν existed, we then can let $\tilde{\nu}$ be the measure obtained from ν by flipping the infinite negative atom and the above discussion would imply that $\tilde{\nu} = \mu$. We now obtain a contradiction by finding an event which has positive μ–measure and 0 $\tilde{\nu}$–measure.

Consider the event E that the origin belongs to an infinite positive atom and the value of the configuration at the origin is M. It easily follows from the construction of $\tilde{\nu}$ and the fact that $-M$ is not an element of the alphabet that this event has $\tilde{\nu}$–measure 0.

We now show $\mu(E) > 0$. Let E_1 be the event that the origin is in an infinite positive atom and let F be the event that the configuration at the origin has value M. Clearly both of these events have positive μ–measure and so we need to show that they are positively correlated. Now let E_1^n be the event that there is a path of length n starting from the origin on which the configuration is positive and such that no two successive points take value 1. Since the set of compatible configurations on Λ_m which take the value M on $\partial\Lambda_m$ is what is called a distributive lattice, it follows from the FKG inequality (see [4]) that if $m > n$ $\mu_m^+(E_1^n \cap F) \geq \mu_m^+(E_1^n)\mu_m^+(F)$. Letting $m \to \infty$ and then $n \to \infty$ (and noting that $\bigcap_{n=1}^\infty E_1^n = E_1$) gives us $\mu(E_1 \cap F) \geq \mu(E_1)\mu(F)$ which is > 0. ∎

Before proving Theorem 1.15, we first give the following lemma (which is analogous to Lemmas 2.6 and 4.1) whose proof we give afterwards.

LEMMA 4.6: *Let μ be a measure of maximal entropy for X with some $k \in \{1, \ldots, M-1\}$. Then for any α with $M\alpha$ an integer and for any finite subset $G \subseteq \mathbf{Z}^d$,*

$$\mu(|\eta(x)| \leq \alpha M \ \ \forall x \in G) \leq (2^{2d+1}\alpha)^{|G|}.$$

Proof of Theorem 1.15: Let $\alpha = \frac{M-k}{M}$ and μ be any measure of maximal entropy. We first want to show that the set $\{x: |\eta(x)| \leq M-k\}$ does not percolate which means that with probability 1 this random set contains no infinite connected subset. Since the number of paths of length n is at most $2d(2d-1)^{n-1}$, Lemma 4.6 implies that the probability that there is a path of length n starting from the origin on which $|\eta(x)| \leq \alpha M$ (or $|\eta(x)| \leq M-k$ by definition of α) is at most

$$2d(2d-1)^{n-1}(2^{2d+1}\alpha)^{n+1}.$$

Since $\frac{M-k}{M} < \frac{1}{(2d-1)2^{2d+1}}$, this probability goes to 0 as $n \to \infty$. Therefore the set $\{x: |\eta(x)| \leq M-k\}$ does not percolate, as desired.

Finally, as $k \leq M-1$ and $-M+k$ is the smallest state, one can proceed by imitating the proof of Theorem 1.17 in [3] to demonstrate uniqueness. We therefore only sketch this argument refering the reader to the above proof for more complete details.

If ν and λ are probability measures on F^S where S is countable (perhaps finite), we say $\nu \preceq \lambda$ if there exists a probability measure m on $F^S \times F^S$ whose first and second marginals are ν and λ respectively and such that

$$m\{(\eta,\delta): \eta(x) \leq \delta(x)\ \forall\, x \in S\} = 1.$$

Let $\mu^+ = \lim_{n\to\infty} \mu_n^+$ where μ_n^+ is defined earlier in this section. Then one can easily show that $\mu \preceq \mu^+$. (This follows from the fact (see Lemma 2.3 in [3]) that if η and δ are configurations defined on $\partial\Lambda_n$ with $\eta(x) \leq \delta(x)\ \forall\, x$, then μ_η and μ_δ (as defined in Definition 2.5) satisfy $\mu_\eta \preceq \mu_\delta$.)

To show $\mu^+ \preceq \mu$, it suffices to show that this holds when the measures are restricted to a fixed Λ_n. Since under μ, the set $\{x: |\eta(x)| \leq M-k\}$ does not percolate, there is a type of surface contour surrounding Λ_n on which the configuration is at least $M-k+1$. Inside this contour, the configuration is uniform over all configurations with these boundary values or equivalently with boundary values M and we obtain $\mu^+ \preceq \mu$ on Λ_n and therefore uniqueness. ∎

Proof of Lemma 4.6: Let $\tilde{\mu}$ be obtained from μ by flipping all the infinite negative atoms. As we have seen, $\tilde{\mu}$ has maximal entropy. It is also clear that $\tilde{\mu}(|\eta(x)| \leq \alpha M\ \ \forall x \in G) = \mu(|\eta(x)| \leq \alpha M\ \ \forall x \in G)$.

Let E denote the event in question and let $\beta = \tilde{\mu}(E)$. We want to show $\beta \leq (2^{2d+1}\alpha)^{|G|}$. If $\beta = 0$, there is nothing to prove and so we assume $\beta > 0$

and let $\epsilon < \beta$. We next take Λ_n containing $G \cup \partial G^c$ with n so large that with probability greater than $1-\epsilon$, any (necessarily finite) negative flippable atom intersecting either G or ∂G^c is contained in Λ_{n-1}. Letting F be this event, we have that $\tilde{\mu}(F) > 1-\epsilon$ and so $\tilde{\mu}(E \cap F) > \beta - \epsilon$. There is then a configuration η with location set $\partial\Lambda_n$ such that $\tilde{\mu}(\eta) > 0$ and $\tilde{\mu}(\ |\eta \text{ on } \partial\Lambda_n)$ gives $E \cap F$ probability $> \beta - \epsilon$. Proposition 1.20 tells us that this conditional measure restricted to Λ_n is uniform distribution μ_η on Ω_η (see Definition 2.5). Note that the events E and F are measurable with respect to the configuration on Λ_n and so we can think of them as subsets of Ω_η.

Next, for each configuration $\xi \in \Omega_\eta \cap E \cap F$, let $\tilde{\xi}$ be the configuration obtained from ξ by reversing the sign of any negative flippable atom which meets $G \cup \partial G^c$ (which necessarily is contained in Λ_{n-1}) and setting $\xi \equiv 1$ on G. Note that $\tilde{\xi} \in \Omega_\eta \cap E \cap F$ and $\tilde{\xi}(x) = 1$ for each $x \in G$. Since

$$|G \cup \partial G^c| \leq (2d+1)|G|,$$

the set $\{\xi' \in \Omega_\eta \cap E \cap F |\ \tilde{\xi}' = \tilde{\xi}\}$ has cardinality at most $2^{(2d+1)|G|}(\alpha M)^{|G|}$. This is because from $\tilde{\xi}$, one can recover ξ up to the sign of the atoms touching $G \cup \partial G^c$ and the αM possible values on G.

Given $\xi \in \Omega_\eta \cap E \cap F$, define the class $C(\tilde{\xi})$ to be

$$\{\rho |\ \rho \text{ is a configuration on } \Lambda_n \\ \text{with } \rho(x) = \tilde{\xi}(x) \text{ for } x \notin G \text{ and } \rho(x) > 0 \text{ for } x \in G\}.$$

Clearly $C(\tilde{\xi}) \subseteq \Omega_\eta \cap F$, $|C(\tilde{\xi})| = M^{|G|}$ and $C(\tilde{\xi}) \cap C(\tilde{\xi}') = \emptyset$ for $\tilde{\xi} \neq \tilde{\xi}'$ since if $\tilde{\xi} \neq \tilde{\xi}'$, they must differ at some point not in G as they are $\equiv 1$ on G.

We therefore obtain

$$\begin{aligned}\beta - \epsilon < \mu_\eta(E \cap F) &= \frac{|\Omega_\eta \cap E \cap F|}{|\Omega_\eta|} \\ &\leq \frac{|\{\tilde{\xi} | \xi \in \Omega_\eta \cap E \cap F\}| 2^{(2d+1)|G|}(\alpha M)^{|G|}}{|\{\tilde{\xi} | \xi \in \Omega_\eta \cap E \cap F\}| M^{|G|}} = (2^{2d+1}\alpha)^{|G|}.\end{aligned}$$

As ϵ is arbitrary, we obtain the desired inequality. ∎

5. Uniqueness via state splitting

In this section, we prove Theorems 1.17 and 1.18. We first mention that the fact that X^n has a unique measure of maximal entropy for large n also follows from the Dobrushin uniqueness condition (Theorem 8.7 in [5]) but our method will be more direct.

Proof of Theorem 1.17: It is obvious that each X^n for $n \geq 1$ is strongly irreducible.

Let G denote the set of n new states which we are adding. Let μ be any measure of maximal entropy for X^n. Since any element of G can sit next to any other element, Proposition 1.20 tells us that for each $x \in \mathbf{Z}^d$,

$$\mu(\eta(x) \in G|\mathcal{F}_x) \geq \frac{n}{n+k} a.s.$$

where $\mathcal{F}_x$ denotes the σ–field generated by $\{\eta(y)\}_{x \neq y \in \mathbf{Z}^d}$. Equivalently,

$$\mu(\eta(x) \notin G|\mathcal{F}_x) \leq \frac{k}{n+k} a.s.$$

It follows that if $\{\eta(x)\}$ has distribution μ, then $\{x \in \mathbf{Z}^d\colon \eta(x) \notin G\}$ is dominated by an i.i.d. process with density $\frac{k}{n+k}$ in the following sense. If $\{\delta(x)\colon x \in \mathbf{Z}^d\}$ is an independent process with $P(\delta(x) = 1) = \frac{k}{n+k}$ and $P(\delta(x) = 0) = \frac{n}{n+k}$ for all x, then $\{\eta(x)\colon x \in \mathbf{Z}^d\}$ and $\{\delta(x)\colon x \in \mathbf{Z}^d\}$ can be defined on the same probability space so that

$$\{x \in \mathbf{Z}^d\colon \eta(x) \notin G\} \subseteq \{x \in \mathbf{Z}^d\colon \delta(x) = 1\}$$

with probability 1.

Now let ν denote the probability measure on $\{0,1\}^{\mathbf{Z}^d}$ corresponding to the process $\{\delta(x)\colon x \in \mathbf{Z}^d\}$. Let μ and $\tilde{\mu}$ be two measures of maximal entropy for X^n. Finally let $E \subseteq X^n \times X^n$ be the set of pairs of configurations (η, η') such there exists an infinite path π in $\mathbf{Z}^d$ such that for all $x \in \pi$, at least one of $\eta(x)$ and $\eta'(x)$ is not in G.

If we can show that $\mu \times \tilde{\mu}(E) = 0$, it will follow from Lemma 2.9 that $\mu = \tilde{\mu}$, proving uniqueness of the measure of maximal entropy. Using the containment above, we obtain

$$\mu \times \tilde{\mu}(E) \leq \nu \times \nu(F)$$

where $F \subseteq \{0,1\}^{\mathbf{Z}^d} \times \{0,1\}^{\mathbf{Z}^d}$ is the set of pairs of configurations (δ, δ') such there exists an infinite path π in $\mathbf{Z}^d$ such that for all $x \in \pi$, at least one of $\delta(x)$ and $\delta'(x)$ is 1.

For each $x \in \mathbf{Z}^d$, the probability that at least one of $\delta(x)$ and $\delta'(x)$ is 1 is $\frac{2k}{n+k} - (\frac{k}{n+k})^2$. Since for different x these are independent and $\frac{2k}{n+k} - (\frac{k}{n+k})^2 < p_c(d)$, it follows that $\nu \times \nu(F) = 0$, as desired. ∎

Proof of Theorem 1.18: Let X denote the subshift with states 1 and -1 where 1 and -1 are not allowed next to each other. This of course implies $|X| = 2$ where the only configurations are all 1's or all -1's. Using the nontrivial fact that $p_c(2) \geq 1/2$ (see [8] and [7]), Theorem 1.17 immediately implies that the "iceberg" model with $M = 1$ in 2 dimensions has a unique measure of maximal entropy.*

Also, the methods of [3] can also be applied to this system to show that for large M it has more than one measure of maximal entropy. The details are left to the reader. ∎

The reader should notice that by Theorem 1.17 if we take the "iceberg" model in any dimension with arbitrary M, then we can still obtain a unique measure of maximal entropy provided we increase the number of zeroes sufficiently.

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* *Note added in proof.* It is not hard to show that if we take 1 (rather than 5) v's in the "iceberg" model, there is a unique measure of maximal entropy without Theorem 1.12. this argument is based on monotonicity and the fact that $p_c(2) > 1/2$ which implies that the 2's do not percolate w.r.t. any measure of maximal entropy.

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